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Abstract

This paper studies systematically how different adaptive expectation rules affect the stability of rational expectation equilibria (REE) in OLG models. We distinguish learning about price levels and inflation rates. When agents form price expectations as an average of past prices, then the monetary steady state tends to be stable. The autarky steady state is unstable, reversing the stability of the two stationary REE under perfect foresight. When adaptive expectations are formed using inflation rates, then both REE can be stable. Roughly, if the current inflation rate is included in the forecasting rule then the autarky REE tends to be stable. When agents are restricted to only use lagged observed inflation, then the monetary REE tends to be stable. We show how the familiar OLS learning fits our results and we demonstrate that different regressions yield different stability results. Finally, we relate our analysis to expectational stability.

1 Introduction

How stable is equilibrium selection via adaptive expectations with respect to the learning specification? This paper studies systematically how different adaptive expectation rules affect the stability of rational expectation equilibria (REE) in dynamic models with multiple REE. While our results are valid for a wide range of dynamic models, we focus particularly on overlapping generations (OLG) models since most of the existing literature is concerned with this class of models. In particular, we find that results of earlier studies (e.g. Lucas (1986) and Marcet and Sargent (1989)) which found that certain adaptive learning rules select the 'good' monetary REE do not hold if agents learn about inflation rates and are allowed to take *current* inflation into account. For this case we show that the 'bad' autarky REE is locally stable for a wide class of adaptive expectations formation. If agents are restricted to *lagged* inflation or if the weight put on current inflation becomes negligible as time goes to infinity, then the autarky REE is unstable. The Marcet and Sargent (1989) OLS learning specification belongs to this class of learning rules. But slightly different OLS regressions yield opposite results. Furthermore, we provide a general result when expectations are formed as a convex combination of past *price levels*.

Dynamic macroeconomic models often have multiple REE. One particularly well-studied class of such models is the class of OLG models following Samuealson (1958) with money and government deficit. Specifically, there are two REE with constant inflation rates. The low inflation rate REE has the usual comparative static properties, while the high inflation REE has 'perverse' comparative statics, as Marcet and Sargent (1989) put it. The low inflation REE is locally unstable under perfect foresight so that most paths converge to the high inflation REE in which money has no value. In other words, perfect foresight selects the 'bad' equilibrium whose properties make little economic sense. An important question is how stability of these REE are affected once the strong assumption of perfect foresight is replaced by adaptive expectation formation. Do adaptive expectations change stability of REE? If so, which adaptive rules lead to stability of one or the other REE? Is it even possible to discard certain REE because no sensible adaptive rule converges to the REE? If that is the case then adaptive expectations might serve as an viable equilibrium selection device. See for example Sargent (1993) for a forceful argument in this direction. These questions have been studied of OLG models for example by Lucas (1986) and Marcet and Sargent (1989). Lucas (1986) shows that if agents forecast prices via the sample average of observed past price levels, then the stability of the two REE is opposite than under perfect foresight. Most price paths converge to the low inflation REE. Marcet and Sargent (1989) confirm Lucas' result when agents use an OLS regression of past price levels (excluding current prices) to forecast next period's inflation. Arifovic (1992) finds that agents learning via a genetic algorithm also converge to the low inflation REE. All of these studies point to the conjecture that the low inflation REE tends to be stable under adaptive expectations while the high inflation REE is not. Since all these papers specify very specific learning rules, an important questions is, how robust these results are.

Instead of using very specific adaptive learning forms, we aim at more general learning specifications. This allows us to pinpoint the most important features of adaptive rules with respect to equilibrium selection. We distinguish between two different classes: learning via averaging past price *levels* and *inflation rates*. We prove a general result which states that when price expectations are in the convex hull of past prices and sufficient weight is put on recent observations, then prices will converge to the low inflation REE. An example of this class is Lucas (1986). But specifying learning on the basis of price levels has some undesirable properties. If prices have been increasing since the initial period, any forecast which lies in the convex hull of these past prices will be below the last price. In other words, agents expect a deflation despite positive inflation in all previous periods. Thus learning about inflation rates seems more appealing. Unfortunately, there are no general results in this case and all of the following results are local to the REE. We study adaptive expectations which use last period's expected inflation and either current or lagged realized inflation as arguments for forecasting next period's inflation: *(i)* $\pi_{t+1}^e = F(\pi_t^e, \pi_t)$ and *(ii)* $\pi_{t+1}^e = F(\pi_t^e, \pi_{t-1})$. It turns out that this timing whether to include current or lagged inflation is crucial for stability. Note, that in general current inflation is in the information set of agents, e.g. when agents are allowed to submit expectation schedules. Hence current inflation can in principle be used to forecast inflation. Some weak restrictions on the form of the function F and its derivatives are made to obtain the results. We find that the high inflation REE is stable under rule *(i)* and unstable under rule *(ii)*. Stability of the low inflation REE is not as clearcut, but if it is stable using specification *(ii)* than it is also stable using specification *(i)*. One important result is thus, that the high inflation REE can be stable with adaptive expectations which is in contrast to the Lucas (1986), Marcet and Sargent (1989) and Arifovic (1992) studies. The key determinant for the stability of the high inflation REE is how much weight is put on current inflation when forecasting tomorrow's inflation. Generalizing the above specifications to allow for time-varying weights, we find that when the weight on current inflation goes to zero as time grows, then the high inflation REE becomes unstable. As an illustration, we demonstrate using the familiar OLS example, that different OLS regressions yield different stability results.

Finally, we compare our stability results to that of expectational stability (E-stability). See Lucas (1978) and DeCanio (1979) for the first definitions of this concept and Evans (1985, 1989) and Evans and Honkapohja (1992, 1993, 1994ab, 1995) for recent results. Instead of learning prices or inflation rates directly, E-stability studies learning the law of motion of the economy in fictitious time. It turns out that only the low inflation REE is E-stable while the high inflation REE is not E-stable. We show that E-stability is closely linked to real time adaptive learning and yields the same stability conditions when the weight put on current inflation in the adaptive real time learning rule goes to zero.

Related literature include Bray (1982) who studies adaptive learning in a cobweb model. Guesnerie and Woodford (1991) study the stability of cycles with adaptive learning rules. The stability of sunspots under learning has been studied by Woodford (1990), Evans (1989) and Evans and Honkapohja (1992). Bertocchi and Wang (1994) look at Bayesian learning in OLG models. Howitt (1992) presents a policy motivated study of interest rate control and learning.

The rest of the paper is organized as follows. Section 2 presents the basic OLG model, defines the REE and studies stability under perfect foresight. Section 3 presents a linear example. Section 4 studies learning using averages of past prices, section 5 presents the results when agents learn inflation rates. Section 6 relates the results to the concept of E-stability and section 7 concludes.

2 The general model

Two general classes of dynamic models can be written as

(1) $$p_t = T(p^e_{t+1})$$

and

(2) $$\pi_t = W(\pi^e_t, \pi^e_{t+1}).$$

In model (1) the variable of interest is p_t. Its current value is only dependent on the one period ahead expectation. OLG models where the dynamics are defined over prices are of this type. However, when the law of motion of the OLG model is written in terms of inflation rates, the current value of inflation depends not only of expectation about next period, but also on inflation expectations from the last period. Hence we need to define the slightly more general model (2). The particular shape of the mapping T and U depend on the concrete model to be studied. They are allowed to be nonlinear. We will focus on models with overlapping generations as concrete examples.

Since there is no uncertainty, agents have perfect foresight and the REE can be written as first-order difference equations

(3) $$p_t = T(p_{t+1})$$

and

(4) $$\pi_t = W(\pi_t, \pi_{t+1}).$$

Stationary REE are fixpoints of these mappings. Relaxing the assumption of perfect foresight, leads us to study agents who form expectations adaptively. The information set of agents at the beginning of period t consists of the entire history of prices including the current price: $H_t = \{p_1, ..., p_t\}$. In general, agents can form their expectations via any measurable time-varying function of H_t:

(5) $$p^e_t = F_t(H_t)$$

and

(6) $$\pi^e_t = F_t(H_t).$$

Given a family of expectations functions F_t, the local stability of the different REE can be determined. We will restrict the class of expectation function later in the paper to allow for a more parsimonious representation. The most prominent example of dynamic models with multiple REE is the class of OLG models. Instead of using the general model presented in this section, we will cast the rest of the paper in the notation of OLG models since the crucial ingrediants of the learning rules can be more easily demonstrated in this notation.

3 An extended example: The OLG model

We study a variant of the basic Samuelson (1958) OLG (consumption-loan) model, in which the money supply may be increased each period to finance a constant government deficit. In this section, we review the details of the model in order to fix notation and properties. See Azariadis (1993) and Sargent (1987) for excellent textbook treatments of OLG models.

There is one perishable good in each period. Generations live two periods, and are indexed $t = 0, 1, \ldots$ by the date of birth. Each generation consists of a single agent (or of identical agents who act identically). Every generation $t > 0$ has the same utility $U(c_{1t}, c_{2t})$, where c_{1t} and c_{2t} are consumption in youth and old-age, respectively, and the same endowment $e_1 > 0$ and $e_2 > 0$ in youth and old-age, respectively.

Forward contracts are not allowed, since it would require trading between agents that are not alive. However, there is a storable good called "money" that has no consumption value. In youth, agents can choose to trade part of their endowment for money. In old-age, agents supply their holding of money inelastically. Normalize the price of money to 1, let p_t be the price of the good in period t, and let $\pi_{t+1} = p_{t+1}/p_t$ be the inflation factor in period $t+1$.

We assume that, when agents trade in their youth, they form point expectations about the price in the next period, in ways yet to be specified. Given a current period price p_t and a price expectation p^e_{t+1}, generation t chooses planned consumption $\langle c_{1t}, c_{2t}\rangle$ and money purchases $m_t \geq 0$ in youth that solve

$$\begin{aligned} \max_{c_{1t}, c_{2t}, m_t} \quad & U(c_{1t}, c_{2t}) \\ \text{subj. to:} \quad & c_{1t} = e_1 - m_t/p_t \\ & c_{2t} = e_2 + m_t/p^e_{t+1}\,. \end{aligned} \tag{7}$$

There is only one degree of freedom in this problem. E.g., the choice of consumption in youth determines the money purchases and the planned consumption in old-age. Furthermore, the optimal consumption levels depend only on the anticipated inflation factor $\pi^e_{t+1} \equiv p^e_{t+1}/p_t$. Let $S(\pi^e_{t+1})$ be the *net supply* of the good in youth, as a function of the expected inflation. Then $m_t = p_t S(\pi^e_{t+1})$.

We impose the following assumptions on S:

Assumption 1

1. *S is differentiable.*
2. *$S'(\cdot) < 0$.*
3. *There is $\pi^a > 1$ such that $S(\pi^a) = 0$.*

The economic significance of these assumptions is as follows:

1. S is differentiable under standard differentiability and concavity assumptions on U.

2. $S'(\cdot) < 0$ if c_1 and c_2 are gross substitutes.

3. π^a is the unique relative price that supports the endowment. We assume $\pi^a > 1$, and hence $S(1) > 0$ so that money is used in equilibrium to transfer consumption from youth to old age. This is often called the "classical" case. This assumption holds, for example, if U is monotone and symmetric and $e_1 > e_2$.

Let $S(0) = \lim_{\pi \to 0} S(\pi)$. Then $S : (0, \pi^a] \to [0, S(0))$ is a diffeomorphism with inverse S^{-1}.

The government issues new money δp_t each period t to purchase $\delta \geq 0$ units of the good, in order to finance a constant real deficit. Such "inflation taxation" is less effective than direct taxation; for example, agents can never be worse off than if they consume their endowments. Therefore, δ cannot be too large. We state now, once and for all, that each result in this paper holds for δ in some interval $[0, \hat{\delta})$, and we will not repeat this restriction each time a result is stated. However, for some results we will explicitly assume that $\delta = 0$ or $\delta > 0$.

In period t, generation $t-1$ supplies its money holdings m_{t-1} inelastically. Hence, the total money supply in period t is:

$$m_t = m_{t-1} + \delta p_t \,. \tag{8}$$

The market for money (and, by Walras' Law, the market for the good) clears when

$$p_t S(\pi^e_{t+1}) = m_t \,. \tag{9}$$

Let $H_t = \{p_s\}_{s=1}^{t}$ be the history of prices up through t. For given parameters δ and m_0, we can write the money supply as a function $m_t = M(H_t)$ of prices. Hence, price expectations are also a function $p^e_{t+1} = P^e_{t+1}(H_t)$ of past and current prices [1]. Fix an expectations rule $P^e \equiv \{P^e_{t+1}(H_t)\}_{t=1}^{\infty}$. Substitute $m_t = M_t(H_t)$ and $p^e_{t+1} = P^e_{t+1}(H_t)$ into the market clearing condition (9), and let $P_t(H_{t-1})$ be the set of prices p_t for which markets clear, given H_{t-1}. $P_t(H_{t-1})$ describes the dynamic behavior of the system for the expectations rule P^e. A price path $H_\infty = \{p_t\}_{t=1}^{\infty}$ is an equilibrium given P^e if, for each t, $p_t \in P_t(H_{t-1})$.

In the autarkic equilibrium, money is expected to have no value and hence has no value. Agents consume their endowments and the government cannot finance its deficit. Our price normalization does not allow us to set the price of money to zero, but we will say that the economy is approximately in autarky when $\pi^e_{t+1} \approx \pi^a$.

For each t, the demand for the good by the government is δ and the demand by the older generation is strictly positive (since they have money to trade). Hence, a necessary condition for equilibrium is that $S(\pi^e_{t+1}) > \delta$, i.e., that $\pi^e_{t+1} < S^{-1}(\delta)$ (for $\delta < S(0)$).

---

[1] The inclusion of the current price in the information set can be defended in at least two ways. First, young agents know how much money the old agents currently hold. They also know their own forecasting rule. Hence they can back out the current equilibrium price. Second, agents also can be allowed to use functional forecasts instead of point forecasts. Thus agents can condition on the current price as noted by Marimon and Sunder (1993).

A price path $\{p_t\}_{t=1}^{\infty}$ is said to be a *rational expectations equilibrium (REE)* if it is an equilibrium for the expectations rule $\{P_{t+1}^e(H_t) = p_{t+1}\}$. By substituting $p_{t+1}^e = p_{t+1}$ into the market-clearing equation (9), we see that the following two equations must hold for each t:

$$
\begin{aligned}
&(10) \qquad & p_{t-1}S(\pi_t) &= m_{t-1} \\
&(11) \qquad & p_t S(\pi_{t+1}) &= m_{t-1} + p_t\delta
\end{aligned}
$$

Combining (10) and (11) yields (12), and rearranging yields (13):

$$
\begin{aligned}
&(12) \qquad & p_{t-1}S(\pi_t) &= p_t S(\pi_{t+1}) - p_t\delta \\
&(13) \qquad & S(\pi_{t+1}) &= S(\pi_t)/\pi_t + \delta\ .
\end{aligned}
$$

We can rewrite (13) as:

$$
(14) \qquad \pi_{t+1} = \Pi(\pi_t) \equiv S^{-1}(S(\pi_t)/\pi_t + \delta)\ .
$$

This is valid for $\pi_t \in (\pi^{\min}(\delta), \pi^{\max}(\delta))$, where $\pi^{\max}(\delta) = S^{-1}(\delta)$ and $\pi^{\min}(\delta)$ is the unique solution to $S(\pi)/\pi + \delta < S(0)$. Note also that $\Pi'(\cdot) > 0$. We can find rational expectations equilibria by fixing p_1 and solving

$$
p_1 = \frac{m_0}{S(\pi_1) - \delta}\ ,
$$

for π_1, and then finding subsequent inflation factors iteratively by $\pi_{t+1} = \Pi(\pi_t)$, as long as $\pi_t \in (\pi^{\min}(\delta), \pi^{\max}(\delta))$. Solving (13) for π_{t+1} given π_t, as long as there is such a solution with $0 < \pi_{t+1} < \pi^{\max}(\delta)$.

An equilibrium is said to be a *stationary REE* if the sequence $\{\pi_{t+1}\}_{t=1}^{\infty}$ is constant. From (13), an inflation factor π is a stationary RE inflation factor if and only if

$$
(15) \qquad S(\pi) = S(\pi)/\pi + \delta\ .
$$

If $\delta = 0$, i.e., the money supply is constant, then either $S(\pi) = 0$ (i.e., $\pi = \pi^a$) or $\pi = 1$. For δ close to 0, there are stationary RE inflation factors close to 1 and π^a, as indicated by the following proposition:

Proposition 1 *There are $\hat{\delta} > 0$ and continuous functions $\pi^*(\delta)$ and $\pi^{**}(\delta)$ defined on $[0, \hat{\delta})$ such that:*

1. *$\pi^*(\delta)$ and $\pi^{**}(\delta)$ are stationary RE inflation factors.*
2. *$\pi^*(0) = 1$ and $\pi^{**}(0) = \pi^a$.*
3. *For $\delta \in (0, \hat{\delta})$: $1 < \pi^*(\delta) < \pi^{**}(\delta) < \pi^a$.*
4. *$\Pi'(\pi^*(\delta)) > 1$ and $0 < \Pi'(\pi^{**}(\delta)) < 1$.*

PROOF: Apply the Implicit Function Theorem to (15), which we rewrite as

$$f(\pi,\delta) \equiv S(\pi) - S(\pi)/\pi - \delta = 0 .$$

Then

$$\frac{\partial f}{\partial \pi} = S'(\pi) - S'(\pi)/\pi + S(\pi)/\pi^2 . \tag{16}$$

The existence and properties (1–3) of π^* and π^{**} follow from:

$$\frac{\partial f}{\partial \pi}(1,0) = S(1) > 0 . \tag{17}$$

$$\frac{\partial f}{\partial \pi}(\pi^a,0) = S'(\pi^a)(1-1/\pi^a) < 0 . \tag{18}$$

Now fix $\delta \in [0,\hat{\delta})$ and differentiate (13) to find $\Pi'(\cdot)$:

$$\begin{aligned} S'(\pi_{t+1})d\pi_{t+1} &= (S'(\pi_t)/\pi_t - S(\pi_t)/\pi_t^2)d\pi_t \\ \Pi'(\pi_t) &= (S'(\pi_t)/\pi_t - S(\pi_t)/\pi_t^2)/S'(\pi_{t+1}) . \end{aligned}$$

From (16) and (17), we have:

$$\begin{aligned} S'(\pi^*) - S'(\pi^*)/\pi^* + S(\pi^*)/\pi^{*2} &> 0 \\ S'(\pi^*)/\pi^* - S(\pi^*)/\pi^{*2} &< S'(\pi^*) \\ (S'(\pi^*)/\pi^* - S(\pi^*)/\pi^{*2})/S'(\pi^*) &> 1 . \end{aligned}$$

Therefore, $\Pi'(\pi^*(\delta)) > 1$. Similarly, (16) and (18) imply that $\Pi'(\pi^{**}(\delta)) < 1$. □

Marcet and Sargent (1989) say that the comparative statics of π^* are "classical" because a increase in the budget deficit increases the stationary inflation factor, whereas the comparative statics of π^{**} are "perverse" because the opposite is true. Furthermore, note that the lower inflation factor Pareto dominates the higher inflation factor (by revealed preference).

These stationary RE equilibria with inflation factors $\pi^*(\delta)$ and $\pi^{**}(\delta)$ will also be stationary equilibria for the other expectations rules we consider. The rest of this paper studies their local stability under the various expectations rules.

We say that a stationary RE inflation factor π is *(locally) stable* if it is a (locally) stable fixed-point of the dynamic equation $\pi_{t+1} = \Pi(\pi_t)$. The local stability properties of the stationary REE follow from $\Pi'(\pi^*(\delta)) > 1$ and $0 < \Pi'(\pi^{**}(\delta)) < 1$, in Proposition 1:

Corollary 1 *$\pi^{**}(\delta)$ is locally stable but $\pi^*(\delta)$ is not.*

Furthermore, if $\pi^*(\delta)$ are the unique stationary REE inflation factors, then there are no equilibrium paths with $\pi_t < \pi^*(\delta)$ (such a path would decrease until $\pi_t < \pi^{\min}(\delta)$) and any equilibrium path with $\pi_t > \pi^*(\delta)$ converges monotonically to $\pi^{**}(\delta)$. Hence, in an economy with perfect foresight agents, most price paths will converge to the high inflation REE whose economic properties are nonsensicle.

4 An example

In this section, we introduce a special case that will appear throughout the rest of the paper. Assume that S is an affine function,

$$S(\pi) \;=\; \alpha_0 - \alpha_1 \pi \ ,$$

with $\alpha_0 > \alpha_1 > 0$. (For example, U is Cobb-Douglas and e_1 is sufficiently larger than e_2.) Then:

1. $\pi^a = \alpha_0/\alpha_1 > 1$.
2. $S'(\pi) = -\alpha_1 < 0$.
3. $\pi^{\max}(\delta) = (\alpha_0 - \delta)/\alpha_1$.
4. $\pi^{\min}(\delta) = (\alpha_0 + \delta)/(\alpha_0 + \alpha_1)$.
5. $\pi^{*(*)}(\delta) = \left(\alpha_0 + \alpha_1 - \delta \pm \sqrt{(\alpha_0 + \alpha_1 - \delta)^2 - 4\alpha_0\alpha_1}\right)/2\alpha_1$.
6. $\Pi(\pi_t) = (\alpha_0 + \alpha_1 - \delta - \alpha_0/\pi_t)/\alpha_1$.

Note that (15) is quadratic, and $\pi^*(\delta)$ and $\pi^{**}(\delta)$ are the only two solutions. Hence, there are no other stationary REE inflation factors.

If $U(c_1, c_2) = c_1c_2$, then $S(\pi) = (e_1 - e_2\pi)/2$. π^* and π^{**} are shown in Figure 1 for $e_1 = 2$, $e_2 = 1$. Figure 2 shows Π for these parameter values and $\delta = 0.04$.

5 Price expectations

In this section, we study adaptive expectations where the price expectation is a weighted average of past prices. If $\delta > 0$, then both stationary REE equilibria have monotonically increasing prices and are not stationary equilibria for such adaptive expectations. Therefore, we assume that $\delta = 0$, so that the low-inflation stationary REE, which has a constant price, is a stationary equilibrium.

When $\delta = 0$, the market clearing equation (9) becomes

$$p_t S(p^e_{t+1}/p_t) \;=\; m_0 \ . \tag{19}$$

The unique solution when $p^e_{t+1} = p_t$ is

$$p^* \;=\; m_0/S(1) \ .$$

This is the constant price of the stationary REE.

For fixed p^e_{t+1}, $p_t S(p^e_{t+1}/p_t)$ is increasing, and hence there is a solution to (19) as long as the maximum value of $p_t S(p^e_{t+1}/p_t)$ is at least m_0. For fixed p_t, $p_t S(p^e_{t+1}/p_t)$ is decreasing in p^e_{t+1}. Therefore, the set of p^e_{t+1} for which there is a solution is an interval $[\bar{p}^e_{t+1}, \infty)$. For fixed p^e_{t+1} in this interval, $p_t S(p^e_{t+1}/p_t)$ is strictly increasing in p_t or is negative. Hence, the solution to (19) is unique. Denote the solution by $P(p^e_{t+1})$. By differentiating (19), one can verify that $P' > 0$

Since $P(\cdot)$ is increasing, $p^e_{t+1}/p_t < 1$ if and only if $p^e_{t+1} > p^*$. Hence, $P(p^e_{t+1})$ is an increasing function that crosses the 45° line at p^*, and $p^e_{t+1}/P(p^e_{t+1})$ converges to π^a as $p^e_{t+1} \to \infty$. In summary:

Proposition 2

1. *For each p^e_{t+1} there is a unique solution $P(p^e_{t+1})$ to $m_0 = m_t(p_t, p^e_{t+1})$.*
2. *P is continuous and strictly increasing.*
3. *The stationary monetary equilibrium price p^* is the unique fixed point of P.*
4. *If $p^e_{t+1} > p^*$, then $p^* < P(p^e_{t+1}) < p^e_{t+1}$.*
5. *If $p^e_{t+1} < p^*$, then $p^e_{t+1} < P(p^e_{t+1}) < p^*$.*
6. $\lim_{n\to\infty} P^n(p) = p^*$ *for all $p > 0$.*

For example, if $S(\pi) = \alpha_0 - \alpha_1\pi$, then P is linear, and $P(p^e_{t+1}) = (m_0 + \alpha_1 p^e_{t+1})/\alpha_0$.

The equation of motion for the REE is $p_t = P^{-1}(p_{t-1})$. From the properties of P, we can see that if $p_t > p^*$, then the path increases monotonically and the inflation factor converges monotonically to π^*. There is no equilibrium path with $p_t < p^*$, because the path would decrease monotonically until there is no equilibrium. Hence, p^* is unstable, and all other equilibrium converge to autarky.

However, when prices are predicted as the average of past prices, the dynamics are inverted. This is the most obvious if we look at the adaptive rule

$$p^e_{t+1} = p_{t-1} \; . \tag{20}$$

Then, starting with a price expectation $p^e_2 = p_0$, we get $p_1 = P(p^e_2)$. This then becomes the next price expectation, and so $p_2 = P(p^e_3) = P^2(p^0)$. For any $t \geq 1$, $p_t = P^t(p^0)$, whereas with perfect foresight expectations $p_t = (P^{-1})^t(p^0)$. The model has the same stationary equilibria, but the stability is also reversed. Most paths converge to p^*, and a single perturbation from the autarkic equilibrium begins a path converging to p^*.

This result can be generalized. Suppose that agents form price expectations using a function $P^e_{t+1}(.)$ of past prices. We impose only a few weak properties on P^e_{t+1}.

Assumption 2 *There are $0 < p^\perp \leq p^\top$ such that P^e_{t+1} lies in the convex hull of $\{p^\perp, p^\top, p_1, \ldots, p_t\}$ for all t.*

Assumption 3 *For all $K \subset \mathbb{R}_+$ and $\tau > 0$, if $p_t \in K$ for all $t \geq \tau$, then the distance $d(P^e_{t+1}(p_1, \ldots, p_t), K)$ between the price expectation and K converges to zero.*

Another way to state the Assumption 2 is that the the price expectation lies between the highest and lowest past prices (including $p^\perp$ and $p^\top$). Any rule that eventually sets the price expectation equal to a convex combination of previous prices and previous price expectations, and never sets the price expectation to 0, satisfies Assumption 2.

Adaptive rule (20) satisfies both assumptions, as does the adaptive rule used in Lucas (1986),

$$\frac{1}{p^c_{t+1}} = \frac{1}{t}\sum_{s=0}^{t-1}\frac{1}{p_s} = \frac{1}{t}\frac{1}{p_{t-1}} + \frac{t-1}{t}\frac{1}{p^c_t}, \tag{21}$$

and the following rule:

$$p^c_{t+1} = \frac{1}{t}\sum_{s=0}^{t-1}p_s = \frac{1}{t}p_{t-1} + \frac{t-1}{t}p^c_t, \tag{22}$$

with p^0 exogenous.

Proposition 3 *Under Assumptions 2 and 3:*

1. $\lim_{t\to\infty} p_t = p^*$.
2. *If* $p^\perp \geq p^*$ *(resp.,* $p^\top \leq p^*$*), then* $p_t \geq p^*$ *(resp.,* $p_t \leq p^*$*) for all* t.
3. *If* p^c_{t+1} *does not depend on* p_t*, then there is a unique equilibrium.*

PROOF: We begin with two lemmas.

Lemma 1 *Under Assumption 3, if there are* $0 < p^\perp \leq p^\top$ *such that* $p^\perp \leq p_t \leq p^\top$ *for all* t*, then* $\lim_{t\to\infty} = p^*$.

PROOF: Suppose that

$$\liminf_{t\to\infty} p_t \geq p^\perp \qquad \text{and} \qquad \limsup_{t\to\infty} p_t \leq p^\top .$$

then Assumption 3 implies that

$$\begin{aligned}\liminf_{t\to\infty} P^c_{t+1}(p_1,\ldots,p_t) &\geq p^\perp \text{, and} \\ \limsup_{t\to\infty} P^c_{t+1}(p_1,\ldots,p_t) &\leq p^\top .\end{aligned}$$

Therefore, since $p_t = P(p^c_{t+1})$ and P is continuous and strictly increasing,

$$\begin{aligned}\liminf_{t\to\infty} p_t &= \liminf_{t\to\infty} P(p^c_{t+1}) \geq P(p^\perp) \\ \limsup_{t\to\infty} p_t &= \limsup_{t\to\infty} P(p^c_{t+1}) \leq P(p^\top)\end{aligned}$$

By induction on n,

$$\liminf_{t\to\infty} p_t \geq P^n(p^\perp) \qquad \text{and} \qquad \limsup_{t\to\infty} p_t \leq P^n(p^\top)$$

for all n. Since $\lim_{n\to\infty} P^n(p) = p^*$ for all $p > 0$, $\lim p_t = p^*$. □

Lemma 2 *Under Assumption 2,*

$$\min\{p^*, p^\perp\} \leq p_t \leq \max\{p^*, p^\top\}$$

for all t.

PROOF: Suppose that $\max\{p_1, p_{t-1}\} \leq \max\{p^*, p^\top\}$. E.g., this is trivially true if $t = 1$. Then either $p^e_{t+1} \leq p^*$ and hence $p_t \leq p^*$, or $p_t < p^e_{t+1}$. In the latter case, Assumption 1 implies that,

$$p^e_{t+1} \leq \max\{p^\top, p_1, \ldots, p_t\} = p^\top .$$

Either way, $p^e_{t+1} \leq \max\{p^*, p^\top\}$ and $p_t \leq \max\{p^*, p^\top\}$. By induction, this is true for all t. A similar argument shows that $p_t \geq \min\{p^*, p^\perp\}$ for all t. □

Now we complete the proof of the proposition:

1. Lemma 2 implies that the condition in Lemma 1 is satisfied, and hence $\lim_{t\to\infty} p_t = p^*$.
2. If $p^\perp \geq p^*$ (resp., $p^\top \leq p^*$), then Lemma 2 implies that $p_t \geq p^*$ (resp., $p_t \leq p^*$) for all t.
3. If P^e_{t+1} does not depend on p_t, the equilibrium in each period is determined uniquely by $p_t = P(P^e_{t+1}(\cdot))$ and price expectations are then uniquely determined by past prices. Hence, there can be only one equilibrium path. (If p^e_{t+1} depends on p_t, then p_t is on both the left and right side of the equation $p_t = P(P^e_{t+1}(p_1, \ldots, p_t))$, and there could be multiple solutions.)

□

The intuition of this result is as follows. Prices can only increase if price expectations are increasing as well. Since agents average past prices to form their expectations of next period's price, they will expect prices do go down even after the price has risen in the past. Thus prices cannot rise forever and inflation has to converge to unity.

6 Inflation expectations

Forming price expectations using a linear combination of past prices has some considerable drawbacks. Suppose that prices have been increasing since the initial period. Then any expectation formed according to a rule satisfying assumptions 1 and 2 will predict that prices will go down next period. Hence, agents expect deflation despite the fact that there has been positive inflation in all previous periods. This unsatisfactory characteristic can be avoided if agents use past inflation rates to form future price expectations. Thus this section studies expectations rules that forecast the ***inflation factor*** as an average of past inflation factors, and then predict $p^e_{t+1} = \pi^e_{t+1} p_t$. From (9), the following must hold for each t:

$$p_{t-1} S(\pi^e_t) = m_{t-1} \tag{23}$$

$$p_t S(\pi^e_{t+1}) = m_{t-1} + p_t \delta \tag{24}$$

Because we are assuming that π^e_{t+1} is a function only of past and current inflation factors, rather than of the absolute price level, we can combine (23) and (24) and rearrange to obtain:

$$\pi_t = \frac{S(\pi^e_t)}{S(\pi^e_{t+1}) - \delta} \,. \tag{25}$$

We assume that a constant inflation factor leads to the same expected inflation factor. Hence, the stationary REE inflation factors are steady-states of (25).

Before considering more general functional forms of forming expectations of future inflation rates, we start with two examples which make the basic mechanism clear. We omit the proofs at this stage since they are just special cases of the propositions which are proved later in this section. In the first example, agents use the *current* inflation rate as expectations of the next period's inflation. The second example specifies inflation expectations as *last period's* realized inflation. Hence, the only difference is the dating of realized inflation which is used to form expectations. It turns out that this timing is a crucial element for local stability of the RE equilibria.

Example 1: $\pi^e_{t+1} = \pi_t$

The law of motion (25) becomes:

$$\pi_t (S(\pi_t) - \delta) = S(\pi_{t-1}) \,. \tag{26}$$

Differentiating:

$$\frac{d\pi_t}{d\pi_{t-1}} = \frac{S'(\pi_{t-1})}{S(\pi_t) - \delta + \pi_t S'(\pi_t)} \,. \tag{27}$$

At $\pi_t = \pi_{t-1} = \pi^{**}(\delta)$, for δ close to zero:

$$\frac{d\pi_t}{d\pi_{t-1}} \approx \frac{S'(\pi^a)}{\pi^a S'(\pi_a)} = 1/\pi^a < 1 \,. \tag{28}$$

Hence, the high-inflation steady state, $\pi^{**}(\delta)$, is locally stable.

At $\pi_t = \pi_{t-1} = \pi^*(\delta)$, for δ close to zero:

$$\frac{d\pi_t}{d\pi_{t-1}} = \frac{S'(1)}{S(1) + S'(1)} \,. \tag{29}$$

Hence, if $S(1) \le -2S'(1)$, $\pi^*(\delta)$ is not locally stable. For example, if $U(c_1, c_2) = c_1 c_2$, then $\pi^*(\delta)$ is not locally stable if $e_1 \le 3e_2$.

Example 2: $\pi^e_{t+1} = \pi_{t-1}$

Suppose instead that $\pi^e_{t+1} = \pi_{t-1}$. Then (25) becomes:

$$f(\pi_t, \pi_{t-1}, \pi_{t-2}) \equiv \pi_t(S(\pi_{t-1}) - \delta) - S(\pi_{t-2}) = 0 \ . \tag{30}$$

Using a linearized form we can show the following:

π^* is locally stable if $-S'(1) < S(1)$. In particular, if π^* is locally stable when $\pi^e_{t+1} = \pi_t$, then π^* is locally stable when $\pi^e_{t+1} = \pi_{t-1}$. For example, $S(\pi) = \alpha_0 - \alpha_1\pi$, then π^* is locally stable if $\alpha_0 < 2\alpha_1$. If $U(c_1, c_2) = c_1c_2$, then π^* is locally unstable if $e_1 \leq 2e_2$. π^{**} is locally unstable

This two examples demonstrate that it is crucial for the stability analysis how the dating of the adaptive expectation formation is specified. If agents are allowed to use current information, then the high inflation equilibrium π^{**} tends to be stable. If agents are restricted to only use information up to period $t-1$ then π^{**} is locally unstable. The stability of the low inflation equilibrium is not as clear cut. In particular, stability depends on the shape of the excess supply function S. However, if π^* is stable when only lagged information is used, then it will also be stable when current inflation rates are included. The reverse is not true. Marimon and Sunder (1993) point out that the inclusion of current information does not make a difference for stability when agents forecast price levels. This results is confirmed by our Proposition 3. However, as the examples show the inclusion of current information in the information set does make a crucial difference when agents forecast inflation rates.

The two examples are expectations rules with short memory. In other words, the adaptive expectations formation only depends either on current inflation or on one period lagged inflation. Inflation lagged more than one period is not included. This will be generalized next. It turns out that it is more convenient to use a recursive formulation including past expected inflation instead past observed inflation directly. But it should be apparent that this specification is fairly general and allows for the entire history of observed past inflation to be relevant for expectation formation. The next two proposition use a expectations specification which is time-invariant, i.e. the functional form is constant over time. This is somewhat restrictive and will be generalized later on.

We consider expectation formation of the following form:

$$\pi^e_{t+1} = F(\pi_t, \pi^e_t) \ . \tag{31}$$

The first example $\pi^e_{t+1} = \pi_t$ is a special case, and we find that our results for this case can be generalized. A second example in this class is the ***constant gain*** updating known in the stochastic approximation literature:

$$\begin{aligned} \pi^e_{t+1} &= (1-\alpha)\pi^e_t + \alpha\pi_t && (32)\\ &= \pi^e_t + \alpha(\pi_t - \pi^e_t) \ , && (33) \end{aligned}$$

with $0 < \alpha < 1$. The intuition is that last period's expectations are updating in the direction of the last observation. The updating rate α determines by how much the expectations are adjusted. Here it is assumed to be constant. We will consider the *decreasing gain*, where α is decreasing with t, later. For the general form (31), we assume that the partial derivatives are (roughly speaking) between 0 and 1, which essentially means that the expectations are revised not too much. In other words, next period's expectations lie locally in the convex hull of this period's expectations and this period's observed inflation. This is reminiscent of the convex hull assumption in the previous section.

Proposition 4 *Assume:*

1. $\pi^e_{t+1} = F(\pi_t, \pi^e_t)$.

2. *F is continuously differentiable, with*

$$0 < \partial F/\partial \pi_t \leq 1 \qquad \text{and} \qquad 0 \leq \partial F/\partial \pi^e_t < 1 \ .$$

3. $\pi = F(\pi, \pi)$ *for all* π.

Then for δ close to but greater than zero, π^{**} is locally stable. Furthermore, if

$$2F_{\pi_t}(1,1)\left(-\frac{S'(1)}{S(1)}\right) < 1 + F_{\pi^e_t}(1,1) \ ,$$

then π^* is locally stable for δ close to zero.

PROOF: Recall that the law of motion for the economy is

$$\pi^e_{t+1} = \frac{S(\pi^e_t)}{S(\pi^e_{t+1}) - \delta}$$

Substitute the law of motion into (31) to get a first-order difference equation for π^e_t:

$$f(\pi^e_{t+1}, \pi^e_t) \ \equiv \ \pi^e_{t+1} - F\left(\frac{S(\pi^e_t)}{S(\pi^e_{t+1}) - \delta}, \pi^e_t\right) = 0.$$

Then $\pi_t \to \pi$ if and only if $\pi^e_t \to \pi$ and the steady states of f are the steady states of the law of motion for π_t. To check local stability, we compute the derivatives of f at the steady states:

$$\begin{aligned} f_{\pi^e_{t+1}}(\pi^e_{t+1}, \pi^e_t) &= 1 + F_{\pi_t}(\pi_t, \pi^e_t)\frac{S(\pi^e_t)S'(\pi^e_{t+1})}{(S(\pi^e_{t+1}) - \delta)^2}, \\ f_{\pi^e_t}(\pi^e_{t+1}, \pi^e_t) &= -F_{\pi_t}(\pi_t, \pi^e_t)\frac{S'(\pi^e_{t+1})}{S(\pi^e_{t+1}) - \delta} - F_{\pi^e_t}(\pi_t, \pi^e_t). \end{aligned}$$

Consider first the stability of π^* with $\delta = 0$:

$$\begin{aligned} f_{\pi^e_{t+1}}(1,1) &= 1 + F_{\pi_t}(1,1)\frac{S'(1)}{(S(1))}, \\ f_{\pi^e_t}(1,1) &= -F_{\pi_t}(1,1)\frac{S'(1)}{S(1)} - F_{\pi^e_t}(1,1). \end{aligned}$$

Thus

$$\frac{d\pi^c_{t+1}}{d\pi^c_t} = -\frac{f_{\pi^c_{t+1}}(1,1)}{f_{\pi^c_t}(1,1)} = \frac{F_{\pi_t}(1,1)\frac{S'(1)}{S(1)} + F_{\pi^c_t}(1,1)}{F_{\pi_t}(1,1)\frac{S'(1)}{(S(1)} + 1}.$$

Hence, the derivative of π^c_{t+1} with respect to π^c_t is less than one in absolute value, implying local stability, if and only if

$$-1 - F_{\pi^c_t} < 2F_{\pi_t}(1,1)\frac{S'(1)}{S(1)}.$$

Note, that the example studied before, $\pi^c_{t+1} = \pi_t$, is just a special case with $F_{\pi_t}(1,1) = 0$ and $F_{\pi^c_t}(1,1) = 1$.

Now, consider the stability of π^{**}. We restrict our attention to $\delta > 0$, because of some singularities in the equations for $\delta = 0$. At $\pi = \pi^{**}$

$$\frac{d\pi^c_{t+1}}{d\pi^c_t} = -\frac{f_{\pi^c_{t+1}}(\pi^{**},\pi^{**})}{f_{\pi^c_t}(\pi^{**},\pi^{**})} = \frac{F_{\pi_t}(\pi^{**},\pi^{**})\frac{S'(\pi^{**})}{S(\pi^{**})-\delta} + F_{\pi^c_t}(\pi^{**},\pi^{**})}{F_{\pi_t}(\pi^{**},\pi^{**})\frac{S'(\pi^{**})}{(S(\pi^{**})-\delta)^2} + 1}. \tag{34}$$

For δ close to zero, $S(\pi^{**}) - \delta$ is close to zero, and hence both numerator and denominator are negative. Thus $\frac{d\pi^c_{t+1}}{d\pi^c_t} < 1$ if and only if

$$F_{\pi^c_t}(\pi^{**},\pi^{**}) - 1 > \left(\frac{F_{\pi_t}(\pi^{**},\pi^{**})S'(\pi^{**})}{S(\pi^{**}) - \delta}\right)\left(\frac{S(\pi^{**})}{S(\pi^{**}) - \delta} - 1\right) \equiv A(\delta)B(\delta).$$

Since $S' < 0$ and $F_{\pi_t}(\pi^{**},\pi^{**}) > 0$ and both are bounded away from zero, we have $\lim_{\delta\to 0} A(\delta) = -\infty$. Furthermore

$$\lim_{\delta\to 0}\frac{S(\pi^{**}(\delta))}{S(\pi^{**}(\delta)) - \delta} = \lim_{\delta\to 0}\frac{S'(\pi^{**}(\delta))\pi^{**\prime}(\delta)}{S'(\pi^{**}(\delta))\pi^{**\prime}(\delta) - 1} > 1$$

from (16). Hence $\lim_{\delta\to 0} B(\delta) > 0$ and $\lim_{\delta\to 0} A(\delta)B(\delta) > -\infty$. Thus π^{**} is locally stable for small positive δ. □

The following proposition shows that the stability of π^* and π^{**} are reversed if agents use *lagged* inflation instead of *current* inflation in their forecasting rule.

Proposition 5 *Assume:*

1. $\pi^c_{t+1} = F(\pi_{t-1}, \pi^c_t)$.

2. *F is continuously differentiable, with*

$$0 < \partial F/\partial\pi_t \leq 1 \qquad \textit{and} \qquad 0 \leq \partial F/\partial\pi^c_t < 1 \ .$$

3. *$\pi = F(\pi,\pi)$ for all π.*

Then for δ close to but greater than zero, π^{**} is locally unstable. Furthermore, if

$$S(1) > -S'(1)\ ,$$

then π^* is locally stable for δ close to or equal to zero.

PROOF: Again, substitute the law of motion into the forecasting rule. This yields a second-order difference equation:

$$f(\pi_t^e, \pi_{t-1}^e, \pi_{t-2}^e) \equiv \pi_t^e - F\left(S(\pi_{t-2}^e)/(S(\pi_{t-1}^e) - \delta), \pi_{t-1}^e\right) = 0$$

The partial derivatives are

$$\begin{aligned} f_{\pi_t^e} &= 1, \\ f_{\pi_{t-1}^e} &= F_{\pi_{t-1}}(.,.)\frac{S(\pi_{t-2}^e)S'(\pi_{t-1}^e)}{(S(\pi_{t-1}^e) - \delta)^2} - F_{\pi_t^e}(.,.), \\ f_{\pi_{t-2}^e} &= -F_{\pi_{t-1}}(.,.)\frac{S'(\pi_{t-2}^e)}{(S(\pi_{t-1}^e) - \delta)}, \end{aligned}$$

where $F_{\pi_{t-1}}$ and $F_{\pi_t^e}$ are the partials of F.

Consider π^* with $\delta = 0$. The characteristic equation for the difference equation linearized around the steady state π^* is

$$x^2 + (-c - F_{\pi_t^e}(1,1))x + c,$$

where $c = -F_{\pi_{t-1}}(1,1)S'(1)/S(1)$. It is easily checked that the roots are lie inside the unit circle if and only if

$$\frac{-S'(1)}{S(1)} < 1.$$

A similar argument as in the proof of the previous proposition shows that π^{**} is unstable for $\delta > 0$. $\square$

These results basically show that results of the simple examples considered earlier hold for more general expectation specifications. Once agents are allowed to use current information in their expectation rule, the high inflation equilibrium tends to be stable and the low inflation equilibrium is less likely to be stable. Note, that there is no obvious economic justification why agents should be excluded from using current inflation to forecast as it is in principle in their information set.

The following two propositions are concerned with functional forms for expectation formation which are allowed to be time-varying. This is important in order to allow *decreasing gain* or more general *variable gain* specifications. We make similar assumptions about the forecasting rule as before. We use the notation F^t for the forecasting rule used in period t.

Assumption 4 *The family of functions $\{F^t(x, y)\}_{t=1}^{\infty}$ has the following properties:*

1. $F^t(x, y) \to F(x, y)$ *as* $t \to \infty$, $\forall x, y$
2. F^t *is continuously differentiable* $\forall t$, *with*

$$0 < \partial F^t/\partial \pi_t \leq 1 \qquad \textit{and} \qquad 0 \leq \partial F^t/\partial \pi_t^e < 1\ .$$

3. $\pi = F^t(\pi, \pi)$ *for all* π, t.

The stability of π^* is not affected by these changes, so we will focus on the stability of π^{**}. The next proposition is slightly different in spirit from the preceding one. Here we fix the deficit δ at a certain level and vary the partial derivatives of F over time. It turns out that if the partial with respect to π_t is getting small enough as t grows, then π^{**} becomes locally unstable.

Proposition 6 *Assume in addition to Assumption 4:*

1. *a fixed* $\delta > 0$
2. $\pi^c_{t+1} = F^t(\pi_t, \pi^c_t)$.
3. $\exists T : F^t_{\pi_t}(\pi^{**}, \pi^{**}) < -\frac{(S(\pi^{**})-b)^2}{S'(\pi^{**})S(\pi^{**})}$ $\forall t > T$

Then π^{**} *is locally unstable.*

PROOF: Recall (34). There the argument was that we can choose δ small enough so that that the numerator and denominator are negative. With fixed δ and $F^t_{\pi_t}(\pi^{**}, \pi^{**})$ small enough, the denominator is positive. Hence the stability condition reverses the sign. It is easily checked that the condition in the proposition give the upper bound of $F^t_{\pi_t}(\pi^{**}, \pi^{**})$ for this to happen. □

An example is

$$\pi^c_{t+1} = (1-\alpha_t)\pi^c_t + \alpha_t \pi_t \quad \text{with} \quad \alpha_t \to 0.$$

When $\alpha_t = 1/t$ next period's expectation is just the average of all past observed inflation rates. Note that these specifications put less weight on current inflation π_t and more weight on π^c_t as t grows. If the weight on π_t becomes small enough, then π^{**} becomes locally unstable. This is intuitive after recalling propositions 4 and 5. There it was shown that the weight which is put on π_t is crucial for stability of π^{**}. The next proposition makes this clear. It shows that if the derivative of F^t with respect to π_t is bounded above 0 for all t, then we can find positive debt levels for which π^{**} is locally stable.

Proposition 7 *Assume in addition to Assumption 4:*

1. $\pi^c_{t+1} = F^t(\pi_t, \pi^c_t)$.
2. $\exists T : F^t_{\pi_t}(\pi_t, \pi^c_t) > \bar{\alpha} > 0$ $\forall t > T$

Then $\exists \bar{\delta} > 0$ *such that* π^{**} *is locally stable for all* $0 < \delta < \bar{\delta}$.

PROOF: The proof is essentially the same as the proof for proposition 4. Since $F^t_{\pi_t}(\pi_t, \pi^{**})$ is bounded above 0, we again can find small enough debt levels δ so that the numerator and denominator are both negative. Then the rest of the proof is analogous. □

An example is a decreasing gain approximation where the weight on π_t does not go to zero:

$$\pi_{t+1}^e = (1-\alpha_t)\pi_t^e + \alpha_t \pi_t \quad \text{with} \quad \alpha_t \to \bar{\alpha} > 0, \alpha_{t+1} > \alpha_t \, .$$

The bottom line of this section is to show that local stability of rational expectations equilibria under adaptive learning will in general depend very much on the exact form how adaptive expectations are specified. The results can be very sensitive to certain ingredients of the learning rule. This is true even for learning specifications which have been studied extensively in the literature. Probably the most known adaptive expectation rule is ordinary least-squares (OLS) learning, see e.g. Marcet and Sargent (1989a,b,c). They have shown that when period-t agents use an OLS regression on past prices including observations up to $t-1$, then π^* is stable and π^{**} is unstable. In our notation we can write this case as

$$\pi_{t+1}^e = \frac{\sum_{s=2}^{t-1} p_s p_{s-1}}{\sum_{s=2}^{t-1} p_{s-1}^2},$$

which is essentially a special case of proposition 5. As we have shown, it is crucial for stability whether date-t information is included. This is also the case here. Thus the revised forecasting rule becomes

$$\pi_{t+1}^e = \frac{\sum_{s=2}^{t} p_s p_{s-1}}{\sum_{s=2}^{t} p_{s-1}^2}$$

which can be rewritten as

$$\pi_{t+1}^e = (1-\alpha_t)\pi_t^e + \alpha_t \pi_t \text{ with } \alpha_t \to \bar{\alpha} > 0.$$

The weights α_t and $\bar{\alpha}$ are implicitly defined by the OLS regression. Hence it is a special case of proposition 7 and therefore π^{**} is locally stable for small enough δ. Another alternative OLS learning specification is to let agents run a regression of past inflation rates on a constant:

$$\pi_{t+1}^e = \frac{(1-t)}{t}\pi_t^e + \frac{1}{t}\pi_t \, ,$$

which is a special case of proposition 6. Thus π^{**} is locally unstable. In other words, stability of REE depends very much on how the adaptive expectations are specified. Different OLS regressions lead to different stability properties. Table 1 summarizes the results studied in this section.

7 Relation to expectational stability

A different but closely related concept of adaptive learning is that of expectational stability (or E-stability). Originally defined by Lucas (1978) and DeCanio (1979), it has been studied extensively by Evans (1985, 1989) and Evans and Honkapohja (1992, 1993, 1994ab, 1995). Instead of learning the price or inflation rate directly, E-stability defines a learning process on the law of motion of the economy. Intuitively, it works as follows. Initially agents possess a given perceived law of motion. Using this perceived law of motion they generate data which can be used to update the perceived law of motion. Under certain

π^c_{t+1}	π^* stable?	π^{**} stable?
π_{t+1}	NO	YES
π_t	$S(1) > -2S'(1)$	YES
π_{t-1}	$S(1) > -S'(1)$	NO
$F(\pi_t, \pi^c_t)$	$\frac{2F_{\pi_t}(1,1)S'(1)}{S(1)} > -1 - F_{\pi^c_t}(1,1)$	YES
$F^t(\pi_t, \pi^c_t)$, $F^t_{\pi^c_t}$ small	$\frac{2F^t_{\pi_t}(1,1)S'(1)}{S(1)} > -1 - F^t_{\pi^c_t}(1,1)$	NO
$F^t(\pi_t, \pi^c_t)$, $F^t_{\pi^c_t}$ large	$\frac{2F^t_{\pi_t}(1,1)S'(1)}{S(1)} > -1 - F^t_{\pi^c_t}(1,1)$	YES
$F(\pi_{t-1}, \pi^c_t)$	$S(1) > -S'(1)$	NO
OLS$(p_{t-1}, .., p_1)\star$	YES	NO
OLS$(p_t, .., p_1)\star$	MAYBE	YES
OLS$(\pi_t, .., \pi_1)\star$	YES	NO

Note: $\star$ valid only for linear model

TABLE 1. Summary

conditions it can be shown that this learning process converges to the true law of motion. In other words, the nonlinear map from the perceived law of motion to the actual law of motion has an asymptotically stable fixed point. If this is the case then the particular law of motion is called expectionally stable (or E-stable). Contrary to the adaptive learning studied previously, this learning iteration takes place in fictitious time. The concept of E-stability can be used to study stability of RE equilibria in OLG models. The condition for E-stability in this context is whether following differential equation has a stable fixed point at π^* or π^{**}:

$$\frac{d\pi}{d\tau} = W(\pi_\tau, \pi_\tau) - \pi_\tau \tag{35}$$

$$\text{with } W(\pi^c_t, \pi^c_{t+1}) = \frac{S(\pi^c_t)}{S(\pi^c_{t+1}) - \delta}. \tag{36}$$

Hence, a fixed point $\hat{\pi}$ is E-stable if

$$|W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})| < 1, \tag{37}$$

where W_1 and W_2 are the partial derivatives. Compare this to the stability condition (34) from the previous section which using the notation (36) yields

$$\left|\frac{d\pi^c_{t+1}}{d\pi^c_t}\right| = \left|\frac{F_{\pi_t}(\hat{\pi}, \hat{\pi})W_1(\hat{\pi}, \hat{\pi}) + F_{\pi^c_t}(\hat{\pi}, \hat{\pi})}{1 - F_{\pi_t}(\hat{\pi}, \hat{\pi})W_2(\hat{\pi}, \hat{\pi})}\right| < 1. \tag{38}$$

Conditions (37) and (38) coincide under the following conditions:

1. $\pi^c_{t+1} = (1 - \alpha_t)\pi^c_t + \alpha_t \pi_t \quad \text{with} \quad \alpha_t \to 0.$
2. $\exists T : 1 - \alpha_t W_2(\hat{\pi}, \hat{\pi}) > 0 \ \forall t > T.$

These conditions are a special case of proposition 6. [2] Hence π^{**} will be unstable whereas π^* might be stable. This also illustrates the connection between adaptive learning in real time and E-stability in fictitious time. These stability criteria only give the same conditions for expectations rules like

$$\pi_{t+1}^e = (1-\alpha_t)\pi_t^e + \alpha_t\pi_t \quad \text{with} \quad \alpha_t \to 0.$$

The connection breaks when too much weight is put on π_t.

8 Conclusion

The purpose of this paper is a systematic study how different adaptive expectations specifications affect the stability of stationary REE in OLG models. In an economy with perfect foresight agents, the autarky REE, which has economically nonsensical properties, is stable, while the sensible monetary REE is unstable. We find that stability for economics with adaptive agents depends very much on certain properties of the expectation formation specification. Hence, it is hard to draw any general conclusion about equilibrium selection via adaptive expectations. Our main results are as follows. Under perfect foresight the low inflation REE is unstable while the high inflation REE is stable. When agents form their expectations as an average of past *price levels* then the stability reverses. The analysis when learning takes place on the basis of *inflation rates* is more complex. When agents are allowed to include *current* inflation into their forecast function, then the high inflation REE tends to be stable for a fairly general class of adaptive learning specifications. The high inflation REE only becomes unstable when the weight put on current inflation becomes negligible over time. When agents are restricted to *lagged* inflation, then the low inflation REE tends to be stable. We study OLS learning as an example of the fragility of stability results with respect to this feature. OLS learning as studied by Marcet and Sargent (1989) uses lagged price levels in the regression resulting in an unstable high inflation REE. Once current prices are included, the high inflation REE becomes locally stable. Furthermore, when a OLS regression of observed inflation rates on a constant is considered, then the high inflation REE is unstable again.

We conclude that in light of these results, adaptive learning is not able to single out one specific REE. Slight changes in the expectation formation rule change the stability of the REE completely. In particular, both REE are locally stable under sensible adaptive rules. Hence, it is doubtful whether adaptive expectations are a sensible equilibrium selection device.

[2] Duffy (1994) shows that the high inflation REE can be E-stable when the excess supply function is increasing in the expected inflation rate.

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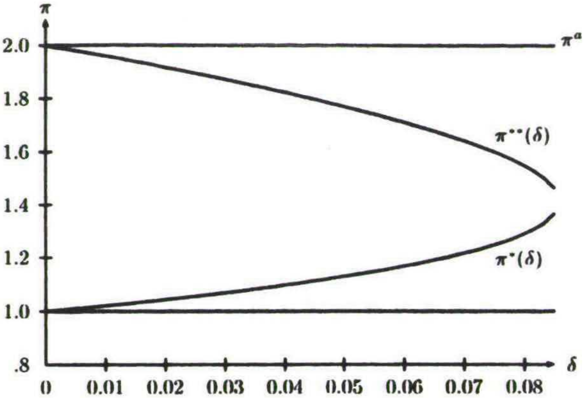

FIGURE 1. The stationary RE equilibrium inflation factors for $S(\pi) = 1 - \pi/2$.

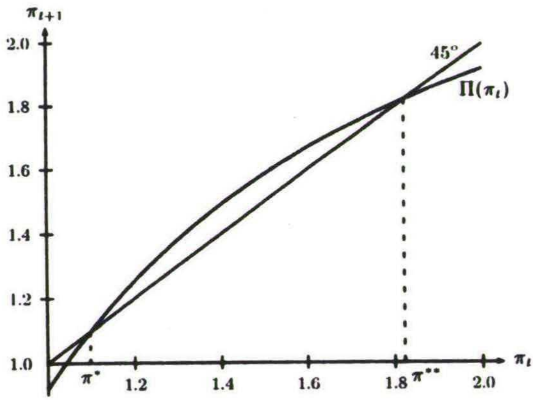

FIGURE 2. The dynamic equation of the RE equilibria for $\delta = 0.04$.

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